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THE DROR-WHITEHEAD THEOREM IN PRO-HOMOTOPY AND SHAPE THEORIES

BY

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ABSTRACT. Many analogues of the classical Whitehead theorem from homotopy theory are now available in pro-homotopy and shape theories. E. Dror has significantly extended the homology version of the Whitehead theorem from the well-known simply connected case to the more general, for instance, nilpotent case. We prove a full analogue of Dror's theorems in pro-homotopy and shape theories. More specifically, suppose $\underline{f}$: $\underline{X} \to \underline{Y}$ is a morphism in the pro-homotopy category of pointed and connected topological spaces which induces isomorphisms of the integral homology pro-groups. Then $\underline{f}$ induces isomorphisms of the homotopy pro-groups, for instance, when $\underline{X}$ and $\underline{Y}$ are simple, nilpotent, complete, or $\underline{H}$-objects; these notions are well known in homotopy theory and we have naturally extended them to pro-homotopy and shape theories.

0. Introduction. The purpose of this paper is to establish an analogue of Dror's generalization of the Whitehead theorem (see **[DR]**), stated below as the Dror-Whitehead theorem, in the context of the pro-homotopy and shape theories. We shall elaborate on these matters in the next few paragraphs.

THE DROR-WHITEHEAD THEOREM. *Suppose a map f: X → Y induces isomorphisms of all the homology groups of spaces X and Y with integral coefficients. Then f induces isomorphisms of all the homotopy groups if and only if* $\Gamma_\omega \pi_* f$ *is an epimorphism,* $\Gamma'_\omega \pi_* f$ *is a monomorphism, and* $\Gamma \pi_* f$ *is a monomorphism.*

The functors Γ_ω, Γ'_ω, and Γ are defined by considering the action of the fundamental group on the homotopy groups; see **[DR]** or §2 of this paper. An important class of spaces to which the Dror-Whitehead theorem applies is nilpotent spaces; see **[DR]**, **[HI]**, **[BK]** or §3 of this paper.

Theorem (4.1.3) of this paper is our extension of the Dror-Whitehead theorem to pro-homotopy and shape theories; also, see Theorems (4.1.1), (4.1.2), (4.1.3), (4.5.1), and Corollary (4.3). Our Corollary (4.3) extends a theorem of Raussen **[RA]** in the same manner as Dror extends the Whitehead theorem. A parallel development of pivotal ingredients of "pro-algebra" is provided by **[SI$_1$]** which extends the work of Stallings **[ST]** and Dror **[DR]** concerning algebra; and our entire program is a natural extension of Dror's work.

As a concluding remark, we may add that many analogues of the various versions of the classical Whitehead theorem have been studied in pro-homotopy

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and shape theories; for instance, see **[AM]**, **[DS]**, **[EH]**, **[MA]**, **[MO]**, **[MR]**, **[RA]** where many other references and related discussions may also be found.

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1. Notation and terminology.

(1.1) *Category theoretic conventions*. A diagram in a category is said to commute if every square or a triangle in the diagram commutes whenever appropriate. All categories will be denoted by script letters. By a map we mean a morphism in a category of interest which will be clear from the context. Suppose $F\colon \mathcal{D} \to \mathcal{E}$ is a functor between the categories $\mathcal{D}$ and $\mathcal{E}$. If $f\colon X \to Y$ is a map in $\mathcal{D}$, we denote the corresponding map by $Ff\colon FX \to FY$, rather than $F(f)\colon F(X) \to F(Y)$, i.e., we omit the cumbersome round brackets whenever convenient. The zero object in any category, whenever it exists, will be denoted by 0. For any category $\mathcal{D}$ we let pro-$\mathcal{D}$ denote the well-known pro-category constructed by Grothendieck **[GR]**. We shall assume familiarity with the construction of pro-$\mathcal{D}$ and related matters; see, for instance, **[AM]**, **[DS]**, **[MA]** for a relevant discussion.

(1.2) *Other conventions*. Our main reference concerning homotopy theory is **[WH]**. We let $\mathcal{Z}$ denote the additive group of integers. The basepoints will often be suppressed. For other related references and terminology, one may also consult **[SI$_1$]**. All the homology groups of spaces are the singular homology groups.

2. Algebraic preliminaries.

(2.1) *Group actions*. Let G and π be groups. We say G is a (left) π-group if there exists a homomorphism $\eta\colon \pi \to \operatorname{Aut} G$ into the group of automorphisms of G, or, equivalently, we say G is a (left) π-group if there exists a map $\alpha\colon \pi \times G \to G$ satisfying $e \cdot g = g$, $(xy) \cdot g = x \cdot (y \cdot g)$, and $x \cdot (gh) = (x \cdot g)(x \cdot h)$, where e is the identity of π, x, y belong to π, g, h belong to G, and $\alpha(x, g)$ is denoted by $x \cdot g$. The phrase "G is a π-group" will be replaced by "G is a π-module" only when G is abelian. Let $\mathcal{G}\mathcal{A}$ denote the category whose objects are the group actions, α's as above, and whose morphisms are pairs $(\phi, \psi)\colon \alpha \to \alpha'$ such that $\phi\colon \pi \times G \to \pi' \times G'$, $\psi\colon G \to G'$, $\psi\alpha = \alpha'\phi$, where α and α' are two objects of $\mathcal{G}\mathcal{A}$.

(2.1.0) *Pro-group actions*. Consider the category pro-$\mathcal{G}\mathcal{A}$ (cf. **[AM]**, **[GR]**). Let $\mathcal{G}$ and pro-$\mathcal{G}$ denote the categories of groups and pro-groups, respectively. Let $\underline{\pi} = (\pi_\lambda, r_{\lambda\lambda'}, \Lambda)$ and $\underline{G} = (G_\lambda, p_{\lambda\lambda'}, \Lambda)$ be two pro-groups. We say $\underline{G}$ is a $\underline{\pi}$-(pro-group) if for each λ we have an action $\alpha_\lambda\colon \pi_\lambda \times G_\lambda \to G_\lambda$ of π_λ on G_λ such that $\underline{\alpha}\colon \underline{\pi} \times \underline{G} \to \underline{G}$ is a special morphism in pro-$\mathcal{G}$; see **[MA]** for a definition of a special morphism and many other related matters concerning pro-$\mathcal{G}$. If G_λ is an abelian group, we shall substitute "$\underline{G}$ is a $\underline{\pi}$-(pro-module)" for "$\underline{G}$ is a $\underline{\pi}$-(pro-group)".

(2.1.1) *π-lower central series of G*. Let G be a π-group. Put $\Gamma_1 G = G$. Define $\Gamma_2 G$ to be the normal π-subgroup of G generated by elements of the form $(x \cdot g)g^{-1}$ for x in π and g in G. Define $\Gamma_\alpha G = \Gamma_2 \Gamma_{\alpha-1} G$ if α is not a limit ordinal and define $\Gamma_\beta G = \bigcap_{\alpha<\beta} \Gamma_\alpha G$ when β is a limit ordinal. This process defines a *lower central series*

(2.1.2) $$\Gamma_s G\colon \cdots \subset \Gamma_{\alpha+1} G \subset \Gamma_\alpha G \subset \cdots \subset \Gamma_2 G \subset \Gamma_1 G = G$$

of G which depends on the action of π on G.

If $\underline{G}$ is a $\underline{\pi}$-(pro-group), we have a pro-(lower central series)

(2.1.3) $$\Gamma_s\underline{G}: \cdots \subset \Gamma_{\alpha+1}\underline{G} \subset \Gamma_\alpha\underline{G} \subset \cdots \subset \Gamma_2\underline{G} \subset \Gamma_1\underline{G} = \underline{G}$$

of $\underline{G}$ which depends on the action of $\underline{\pi}$ on $\underline{G}$.

(2.1.4) *The π-completions of G.* Let β be a limit ordinal. *The π-completion $\hat{\Gamma}_\beta G$ of G up to β* is the inverse limit of the inverse system

(2.1.5) $$\cdots \to G/\Gamma_\alpha G \to \cdots \to G/\Gamma_3 G \to G/\Gamma_2 G \to 0$$

where $\alpha < \beta$. The quotient maps of the form $G \to G/\Gamma_\alpha G$ determine a map i: $G \to \hat{\Gamma}_\beta G$ whose kernel is $\Gamma_\beta G$ and whose cokernel is denoted by $\Gamma'_\beta G = \hat{\Gamma}_\beta G/iG$. We have an exact sequence

(2.1.6) $$0 \to \Gamma_\beta G \to G \xrightarrow{i} \hat{\Gamma}_\beta G \to \Gamma'_\beta G \to 0$$

where $\Gamma'_\beta G$ is a pointed set if G is nonabelian.

Given a $\underline{\pi}$-(pro-group) $\underline{G}$. *The $\underline{\pi}$-completion $\hat{\Gamma}_\beta\underline{G} = (\hat{\Gamma}_\beta G_\lambda, \hat{\Gamma}_\beta p_{\lambda\lambda'}, \Lambda)$ of $\underline{G}$ up to a limit ordinal β* is defined as the (component-wise) inverse limit of

(2.1.7) $$\cdots \to \underline{G}/\Gamma_\alpha\underline{G} \to \cdots \to \underline{G}/\Gamma_3\underline{G} \to \underline{G}/\Gamma_2\underline{G} \to 0$$

where $\alpha < \beta$. We have an exact sequence

(2.1.8) $$0 \to \Gamma_\beta\underline{G} \to \underline{G} \xrightarrow{\underline{i}} \hat{\Gamma}_\beta\underline{G} \to \hat{\Gamma}'_\beta\underline{G} \to 0$$

where $\Gamma'_\beta\underline{G}$ is, in general, a (pointed) pro-set. It is easy to see that $\Gamma_\beta\underline{G}$ is the kernel of $\underline{i}$ in pro-$\mathcal{G}$ and $\Gamma'_\beta\underline{G}$ is the cokernel of $\underline{i}$ if each G_λ is abelian.

(2.2) *The maximal π-perfect subgroup of G.* A π'-group H is *π'-perfect* if $\Gamma_2 H = H$. Every π-group G contains a unique maximal π-perfect subgroup ΓG which is the π-subgroup of G generated by the family of π-perfect subgroups. Observe that $\Gamma G \subset \Gamma_\beta G$ for any ordinal β. Also, Γ can be viewed as a functor from $\mathcal{G}\,\mathcal{A}$ into $\mathcal{G}$.

Given a $\underline{\pi}$-(pro-group) $\underline{G}$. The pro-group $\Gamma\underline{G} = (\Gamma G_\lambda, \Gamma p_{\lambda\lambda'}, \Lambda)$ will be called *the maximal $\underline{\pi}$-perfect pro-subgroup of $\underline{G}$.*

(2.3) *The functoriality.* Observe that Γ_s, Γ_β, Γ'_β, and $\hat{\Gamma}_\beta$ are functors defined on $\mathcal{G}\,\mathcal{A}$: Our constructions $\Gamma_s\underline{G}$, $\Gamma_\beta\underline{G}$, $\Gamma'_\beta\underline{G}$, and $\hat{\Gamma}_\beta$ are the pro-functors defined on pro-$\mathcal{G}\,\mathcal{A}$ which are the natural extensions of these functors.

3. Nilpotent spaces and related matters.

(3.1) *The action of $\pi_1 X$.* All spaces considered in the sequel are pointed and the basepoint is often suppressed. It is well known that the fundamental group $\pi_1 X$ of a space X acts on the homotopy group $\pi_i X$ where $i = 1, 2, \ldots$; and furthermore, this action is natural (cf. [WH]). By "$\pi_i X$ is a $\pi_1 X$-group" we shall always mean this natural action and we often leave the map, $\pi_1 X \times \pi_i X \to \pi_i X$, determining this action unlabelled.

(3.1.1) *The case when $i = 1$.* The group $\pi_1 X$ is considered as a $\pi_1 X$-group, and this action is by conjugation (cf. [WH]); in this case, one obtains the classical lower central series $\Gamma_s \pi_1 X$.

(3.1.2) *The case when $i \geq 2$.* Since $\pi_i X$ is abelian, we often say "$\pi_i X$ is as $\pi_1 X$-module" or we regard $\pi_i X$ as a module over the integral group ring $\mathfrak{Z}\pi_1 X$. The filteration of $\Gamma_s \pi_i X$ is induced by the powers, $(I\pi_1 X)^\alpha$, of the augmentation ideal $I\pi_1 X$ of $\mathfrak{Z}\pi_1 X$ for finite α and this can be appropriately interpreted for infinite α.

(3.2) *A pro-homotopy category.* Let $\mathcal{HT}_0$ denote the homotopy categories of pointed and connected topological spaces. We are interested in the category $\mathcal{C}$ = pro-$\mathcal{HT}_0$. Let $\underline{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be an object of $\mathcal{C}$. Now, $\pi_i\underline{X}$ for $1 \leqslant i < \infty$ is a $\pi_1\underline{X}$-(pro-group).

(3.2.1) *The case when* $i = 1$. We obtain the pro-(lower central series) $\Gamma_s\pi_1\underline{X} = (\Gamma_s\pi_1X_\lambda, \Gamma_s p_{\lambda\lambda'}, \Lambda)$. Observe that the sequence (2.1.8) and $\Gamma\pi_1\underline{X}$ (see (2.2)) are meaningfully defined in this setting.

(3.2.2) *The case when* $i \geqslant 2$. The (abelian) pro-group $\pi_i\underline{X}$ is a $\pi_1\underline{X}$-module and we obtain the lower central series $\Gamma_s\pi_i\underline{X}$, $\Gamma\pi_i\underline{X}$, and the exact sequence (2.1.8).

Observe that $\Gamma_s\pi_i\underline{X}$, $\Gamma\pi_i\underline{X}$, $\Gamma_\beta\pi_i\underline{X}$, $\Gamma'_\beta\pi_i\underline{X}$, and $\hat{\Gamma}_\beta\pi_i\underline{X}$ are functors for $i \geqslant 1$; hence, they are invariants of the isomorphism class of $\underline{X}$ in $\mathcal{C}$.

(3.2.3) *Nilpotent objects in* $\mathcal{C}$. An object $\underline{X}$ in $\mathcal{C}$ is *nilpotent in* $\mathcal{C}$ if for each $i \geqslant 1$ there exists an integer $k = k(i)$ such that $\Gamma_k\pi_i\underline{X} \approx 0$ in pro-$\mathcal{G}$. An object $\underline{X}$ in $\mathcal{C}$ is called *simple in* $\mathcal{C}$ if $\Gamma_2\pi_i\underline{X} \approx 0$ in pro-$\mathcal{G}$ for all $i \geqslant 1$.

(3.2.4) *Complete objects in* $\mathcal{C}$. An object $\underline{X}$ is called *complete in* $\mathcal{C}$ if the natural map $\pi_i\underline{X} \to \hat{\Gamma}_\omega\pi_i\underline{X}$ is an isomorphism in pro-$\mathcal{G}$ for $i \geqslant 1$.

(3.3) *Some preliminary lemmas.* We assume familiarity with the matters related to the "singular functor" and the "geometric realization functor"; see, for instance, [**AM**], [**BK**], [**WH**]. Let $\underline{f}$: $\underline{X} \to \underline{Y}$ be a special map in $\mathcal{C}$. We are only interested in the maps induced by $\underline{f}$ on the homotopy and homology pro-groups; hence, we shall assume without loss of generality that Postnikov decompositions and related constructions can be appropriately applied; see (3.3.2) below and [**AM**, Chapter 1]. The following extends Lemma (6.1) of [**DR**].

(3.3.1) LEMMA. *In addition, suppose* $\pi_i\underline{f}$: $\pi_i\underline{X} \to \pi_i\underline{Y}$ *is an isomorphism for* $0 \leqslant i \leqslant (n-1)$, *and suppose that* $H_i\underline{f}$: $H_i\underline{X} \to H_i\underline{Y}$ *is an isomorphism for* $i = n$ *and an epimorphism for* $i = n + 1$. *Then the following conclusions hold*:

(C_1) *In the case* $n = 1$, $H_i\underline{f}$: $H_i\underline{X} \to H_i\underline{Y}$ *is an isomorphism for* $i = n$ *and an epimorphism for* $i = n + 1$ (*this conclusion merely restates the hypothesis*).

(C_n: $n \geqslant 2$) *In the case* $n \geqslant 2$, *the map* $H_j(\pi_1\underline{X}; \pi_n\underline{X}) \to H_j(\pi_1\underline{Y}; \pi_n\underline{Y})$ *induced by* $\underline{f}$ *is an isomorphism for* $j = 0$ *and an epimorphism for* $j = 1$.

(3.3.2) *Some preliminary discussions.* For each CW-complex X the coskeleton, P_nX, is a functor defined on the homotopy category of CW-complexes [**AM**], [**PO**]. The following is a characterization of P_nX: The homotopy groups of P_nX, $n \geqslant 1$, vanish in dimension $> n$ and the canonical map $X \to P_nX$ is universal with respect to maps of X into CW-complexes with vanishing homotopy groups in dimensions $> n$; see [**AM**]. Observe that Artin and Mazur [**AM**] denote P_nX by $\cos k_{n+1}X$.

(3.3.3) *Proof of Lemma* (3.3.1): *A sketch.* Consider the system $K(\pi_n\underline{X}, n) \to P_n\underline{X} \to P_{n-1}\underline{X}$ of fibrations of the nth stage Postnikov system, i.e., for each λ in Λ, we have a fibration $K(\pi_nX_\lambda, n) \to P_nX_\lambda \to P_{n-1}X_\lambda$. Consider the pro-(spectral sequence)

$$(3.3.4) \qquad \underline{E}^2_{p,q} = \left\{{}_\lambda E^2_{p,q} = H_p\left(P_{n-1}X_\lambda; \left\{H_qK(\pi_nX_\lambda, n)\right\}\right) \Rightarrow H_{p+q}P_nX_\lambda\right\}_\lambda,$$

where the bracket around $H_qK(\pi_n X_\lambda, n)$ signifies that the homology is with "twisted coefficients", i.e., $H_qK(\pi_n X_\lambda, n)$ is considered as a $\pi_1 P_{n-1}X_\lambda$-module. It follows that for each λ in Λ, the sequence

$$(3.3.5)\quad \begin{cases} H_{n+2}P_nX_\lambda \to H_{n+2}P_{n-1}X_\lambda \to H_1(\pi_1 X_\lambda; \pi_n X_\lambda) \to H_{n+1}P_nX_\lambda \\ \qquad \to H_{n+1}P_{n-1}X_\lambda \to H_0(\pi_1 X_\lambda; \pi_n X_\lambda) \to H_nX_\lambda \to H_nP_{n-1}X_\lambda \to 0 \end{cases}$$

is exact.

Observe that for each λ in Λ, $H_{n+1}X_\lambda \to H_{n+1}P_nX_\lambda$ is an epimorphism; hence, $H_{n+1}\underline{X} \to H_{n+1}P_n\underline{X}$ is an epimorphism. It follows that $H_{n+1}P_n\underline{X} \to H_{n+1}P_n\underline{Y}$ is an epimorphism since $H_{n+1}\underline{X} \to H_{n+1}\underline{Y}$ is an epimorphism.

Since (3.3.5) is exact, the corresponding sequences of pro-groups for $\underline{X}$ and $\underline{Y}$ are exact. Form a diagram whose rows are these exact sequences of pro-groups for $\underline{X}$ and $\underline{Y}$ and whose vertical maps are induced by $\underline{f}$. Our proof is finished by the "Five Lemma"; see (3.6.6). □

(3.4) LEMMA. *Suppose* $\underline{f}$: $\underline{X} \to \underline{Y}$ *satisfies the hypotheses of Lemma* (3.3.1). *Then the following conclusion holds*:

($\hat{C}_n$: $n \geqslant 1$) *The induced map* $\hat{\Gamma}_\omega \pi_n \underline{f}$: $\hat{\Gamma}_\omega \pi_n \underline{X} \to \hat{\Gamma}_\omega \pi_n \underline{Y}$ *is an isomorphism.*

PROOF. Observe that $n \geqslant 1$ is some fixed integer. We assume (C_1) or (C_n: $n \geqslant 2$) holds: In either case, we conclude that the map $\pi_n\underline{X}/\Gamma_i\pi_n\underline{X} \to \pi_n\underline{Y}/\Gamma_i\pi_n\underline{Y}$, induced by $\underline{f}$ is an isomorphism. This follows from Theorem (3.3.1) or Theorem (4.2) of [$\mathbf{SI_1}$], respectively. This suffices to prove the result. □

(3.4.1) REMARK. In the setting described above, for each $n \geqslant 1$, the conclusion (C_n) implies ($\hat{C}_n$).

(3.5) LEMMA. *Suppose* $\underline{f}$: $\underline{X} \to \underline{Y}$ *satisfies the hypotheses of Lemma* (3.3.1). *Then the following assertions are equivalent*:

(a) *The map* $\pi_n\underline{f}$ *is an isomorphism.*

(b) $\Gamma_\omega\pi_n\underline{f}$ *is an epimorphism,* $\Gamma'_\omega\pi_n\underline{f}$ *is a monomorphism, and* $\Gamma\pi_n\underline{f}$ *is a monomorphism.*

PROOF. It suffices to show that (b) implies (a). Consider the commutative diagram

$$(3.5.1)\qquad \begin{array}{ccccccccc} 0 & \to & \Gamma_\omega\pi_n\underline{X} & \to & \pi_n\underline{X} & \to & \hat{\Gamma}_\omega\pi_n\underline{X} & \to & \Gamma'_\omega\pi_n\underline{X} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \Gamma_\omega\pi_n\underline{Y} & \to & \pi_n\underline{Y} & \to & \hat{\Gamma}_\omega\pi_n\underline{Y} & \to & \Gamma'_\omega\pi_n\underline{Y} & \to & 0. \end{array}$$

The unlabelled maps $\Gamma_\omega\pi_n\underline{f}$, $\hat{\Gamma}_\omega\pi_n\underline{f}$, and $\Gamma'_\omega\pi_n\underline{f}$ are an epimorphism, an isomorphism, and a monomorphism, respectively. It follows from a "Five Lemma" (this is discussed in (3.6)) that $\pi_n\underline{f}$ is an epimorphism. In the next few paragraphs, we shall show that $\pi_n\underline{f}$ is a monomorphism.

It is easy to see that $\pi_n\underline{X}/\Gamma_\omega\pi_n\underline{X} \to \pi_n\underline{Y}/\Gamma_\omega\pi_n\underline{Y}$ is an epimorphism since $\pi_n\underline{X} \to \pi_n\underline{Y}$ is an epimorphism. It follows from (3.5.1) that both $\pi_n\underline{X}/\Gamma_\omega\pi_n\underline{X} \to \hat{\Gamma}_\omega\pi_n\underline{X}$ and $\pi_n\underline{Y}/\Gamma_\omega\pi_n\underline{Y} \to \hat{\Gamma}_\omega\pi_n\underline{Y}$ are monomorphisms, and $\hat{\Gamma}_\omega\pi_n\underline{f}$ is an isomorphism by Lemma (3.4). It is now easy to see that the map $\pi_n\underline{X}/\Gamma_\omega\pi_n\underline{X} \to \pi_n\underline{Y}/\Gamma_\omega\pi_n\underline{Y}$ is a monomorphism; hence, it is an isomorphism. By Lemma (3.3.2) of

[$\mathbf{SI_1}$] and the argument given above at the limit ordinals, the induced map $\pi_n \underline{X}/\Gamma_\alpha \pi_n \underline{X} \to \pi_n \underline{Y}/\Gamma_\alpha \pi_n \underline{Y}$ is an isomorphism for any ordinal α.

Observe that for cardinality reasons, there exists an ordinal α such that $\Gamma\pi_n \underline{X} = \Gamma_\alpha \pi_n \underline{X}$. Consider a diagram of pro-groups whose rows are $0 \to \Gamma\pi_n \underline{X} \to \pi_n \underline{X} \to \pi_n \underline{X}/\Gamma\pi_n \underline{X} \to 0$ and consider a similar one for $\underline{Y}$. The vertical maps of this diagram are induced by $\underline{f}$. By the "Five Lemma" (see (3.6.6)) it follows that $\pi_n \underline{f}$ is a monomorphism; hence, $\pi_n \underline{f}$ is a bimorphism. This proves $\pi_n \underline{f}$ is an isomorphism; see [**MA**]. □

(3.6) *The setting for a "Five Lemma"*. Given the following diagram

$$\begin{array}{ccccccccccc} 0 & \to & \underline{H} & \overset{\underline{j}}{\to} & \underline{G} & \overset{\underline{i}}{\to} & \underline{K} & \overset{\underline{f}}{\to} & \underline{L} & \to & 0 \\ & & \downarrow \underline{\alpha} & & \downarrow \underline{\beta} & & \downarrow \underline{\gamma} & & \downarrow \underline{\delta} & & \\ 0 & \to & \underline{H}' & \overset{\underline{j}'}{\to} & \underline{G}' & \overset{\underline{i}'}{\to} & \underline{K}' & \overset{\underline{f}'}{\to} & \underline{L}' & \to & 0 \end{array} \tag{3.6.1}$$

such that for each λ in Λ we have a commutative diagram

$$\begin{array}{ccccccccccc} 0 & \to & H_\lambda & \overset{j_\lambda}{\to} & G_\lambda & \overset{i_\lambda}{\to} & K_\lambda & \overset{f_\lambda}{\to} & L_\lambda & \to & 0 \\ & & \downarrow \alpha_\lambda & & \downarrow \beta_\lambda & & \downarrow \gamma_\lambda & & \downarrow \delta_\lambda & & \\ 0 & \to & H'_\lambda & \overset{j'_\lambda}{\to} & G'_\lambda & \overset{i'_\lambda}{\to} & K'_\lambda & \overset{f'_\lambda}{\to} & L'_\lambda & \to & 0 \end{array} \tag{3.6.2}$$

such that each row of (3.6.2) is exact; furthermore, the maps $\underline{\alpha}$, $\underline{\beta}$, and $\underline{\gamma}$ are maps of pro-groups in pro-$\mathcal{G}$, and $\underline{\delta}$ is a map of pro-(pointed sets). More specifically, $L_\lambda = K_\lambda / i_\lambda G_\lambda$ and $L'_\lambda = K'_\lambda / i'_\lambda G'_\lambda$ for each λ in Λ; and the maps f_λ and f'_λ are the quotient maps. Notation:

$$\underline{G} = (G_\lambda, p_{\lambda\lambda'}, \Lambda), \quad \underline{G}' = (G'_\lambda, p'_{\lambda\lambda'}, \Lambda), \quad \underline{K} = (K_\lambda, q_{\lambda\lambda'}, \Lambda),$$
$$\underline{K}' = (K'_\lambda, q'_{\lambda\lambda'}, \Lambda), \quad \underline{H} = (H_\lambda, p_{\lambda\lambda'}, \Lambda), \quad \underline{H}' = (H'_\lambda, p'_{\lambda\lambda'}, \Lambda),$$
$$\underline{L} = (L_\lambda, r_{\lambda\lambda'}, \Lambda) \quad \text{and} \quad \underline{L}' = (L'_\lambda, r'_{\lambda\lambda'}, \Lambda).$$

Moreover, for each λ in Λ, the homomorphisms j_λ and j'_λ are inclusion; and, therefore, we have again denoted the restrictions of the homomorphisms $p_{\lambda\lambda'}$ and $p'_{\lambda\lambda'}$ by $p_{\lambda\lambda'}$ and $p'_{\lambda\lambda'}$.

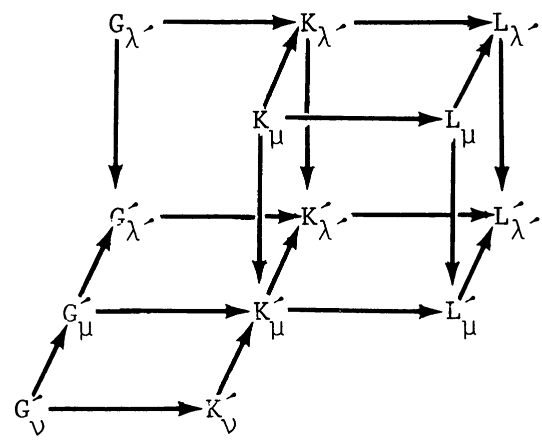
(3.6.3) LEMMA ("A FIVE LEMMA"). *In the setting*: *If $\underline{\alpha}$ is an epimorphism, $\underline{\gamma}$ is an epimorphism, and $\underline{\delta}$ is a monomorphism, then $\underline{\beta}$ is an epimorphism.*

PROOF. Given an index λ in Λ. (I) Since $\underline{\alpha}$ is an epimorphism, there exists $\lambda' \geqslant \lambda$ such that the image $\mathrm{Im}(H'_{\lambda'} \to H'_\lambda)$ is contained in the image $\mathrm{Im}(H_\lambda \to H'_\lambda)$. (II) Given $\lambda' \geqslant \lambda$ as asserted, choose $\mu \geqslant \lambda'$ such that the induced map between the kernels, $[\mathrm{Ker}(L_\mu \to L'_\mu) \to \mathrm{Ker}(L_{\lambda'} \to L'_{\lambda'})]$, is the zero map; this follows since $\underline{\delta}$ is a monomorphism. (III) Choose $\nu \geqslant \mu$ such that $\mathrm{Im}(K'_\nu \to K'_\mu)$ is contained in $\mathrm{Im}(K_\mu \to K'_\mu)$.

Claim. The image $\mathrm{Im}(G'_\nu \to G'_\lambda)$ is contained in the image $\mathrm{Im}(G_\lambda \to G'_\lambda)$.

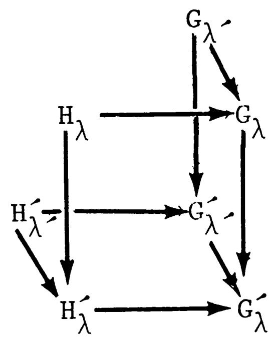
We next prove this claim by considering the following commutative diagrams:

(3.6.4)



and

(3.6.5)



Suppose an element g_ν' of G_ν' goes to an element $g_{\lambda'}'$ in $G_{\lambda'}'$ in (3.6.4). By a chase of (3.6.4) along with utilizing (II) and (III), we conclude that there exists an element $g_{\lambda'}$ in $G_{\lambda'}$ whose image $x_{\lambda'}'$ in $G_{\lambda'}'$ has the property that $(x_{\lambda'}')^{-1}g_{\lambda'}$ belongs to $H_{\lambda'}'$. Now chase the diagram (3.6.4) and use (I) to conclude that there exists an element h_λ in H_λ whose image g_λ' in G_λ' equals the image $(x_\lambda')^{-1}g_\lambda'$ in G_λ' of $(x_{\lambda'}')^{-1}g_{\lambda'}'$. Hence, the elements x_λ' and $(x_\lambda')^{-1}g_\lambda'$ are in $\operatorname{Im}(G_\lambda \to G_\lambda')$. This means $g_\lambda' = x_\lambda'(x_\lambda')^{-1}g_\lambda'$ is in $\operatorname{Im}(G_\lambda \to G_\lambda')$. This proves our claim and suffices to prove that $\underline{\beta}$ is an epimorphism (cf. [**MA**]). □

(3.6.6) *Remarks on the "Five Lemma"*. The category of pro-(abelian groups) is abelian [**AM**]; hence, one may assume the "Five Lemma" while working in this category. In pro-$\mathcal{G}$, the various versions of this lemma require some proof: Although we have not studied all the possible versions, we have proved (for lack of better reference concerning these matters) a "Weak Five Lemma" in pro-$\mathcal{G}$ (see [**SI$_1$**]), along with the result of (3.6) which suffices for our applications.

For convenience of reference, we shall summarize the main results of this section in the following section.

4. A summary of the main results.

(4.1) *The setting*. Throughout the following we let $f\colon \underline{X} \to \underline{Y}$ denote a map in $\mathcal{C}$. We say $\underline{f}$ is a *weak pro-homotopy equivalence* if $\underline{f}$ induces isomorphisms of all homotopy pro-groups of $\underline{X}$ and $\underline{Y}$. All homology is considered with coefficients in $\mathfrak{Z}$. The map $\underline{f}$ is called a *pro-homology equivalence* if $\underline{f}$ induces isomorphisms of all the homology pro-groups of $\underline{X}$ and $\underline{Y}$. We have proved the following results.

(4.1.1) THEOREM. *Given a map* $\underline{f}\colon \underline{X} \to \underline{Y}$ *in* $\mathcal{C}$. *Suppose* (a) $\pi_i \underline{f}\colon \pi_i \underline{X} \to \pi_i \underline{Y}$ *is an isomorphism for* $0 \leqslant i \leqslant (n-1)$, (b) *the induced map* $H_n \underline{f}\colon H_n \underline{X} \to H_n \underline{Y}$ *of the homology pro-groups is an isomorphism, and* (c) *the induced map* $H_{n+1}\underline{f}\colon H_{n+1}\underline{X} \to H_{n+1}\underline{Y}$ *is an epimorphism. Then the induced map* $\hat{\Gamma}_\omega \pi_n \underline{f}\colon \hat{\Gamma}_\omega \pi_n \underline{X} \to \hat{\Gamma}_\omega \pi_n \underline{Y}$ *is an isomorphism.*

(4.1.2) THEOREM. *Suppose* $\underline{f}\colon \underline{X} \to \underline{Y}$ *satisfies the hypotheses of Theorem* (4.1.1), *and suppose* $\underline{f}$ *satisfies the following additional hypotheses*:

(a) $\Gamma_\omega \pi_n \underline{f}\colon \Gamma_\omega \pi_n \underline{X} \to \Gamma_\omega \pi_n \underline{Y}$ *is an epimorphism,*

(b) $\Gamma'_\omega \pi_n \underline{f}\colon \Gamma'_\omega \pi_n \underline{X} \to \Gamma'_\omega \pi_n \underline{Y}$ *is a monomorphism, and*

(c) $\Gamma \pi_n \underline{f}\colon \Gamma \pi_n \underline{X} \to \Gamma \pi_n \underline{Y}$ *is a monomorphism.*

Then $\pi_n \underline{f}\colon \pi_n \underline{X} \to \pi_n \underline{Y}$ *is an isomorphism.*

(4.1.3) THEOREM. *Suppose* $\underline{f}\colon \underline{X} \to \underline{Y}$ *is a pro-homology equivalence*; *see* (4.1) *for terminology. Then the following are equivalent*:

(a) $\underline{f}$ *is a weak pro-homotopy equivalence.*

(b) *For all* i, $1 \leqslant i < \infty$, $\Gamma_\omega \pi_i \underline{f}$ *is an epimorphism,* $\Gamma'_\omega \pi_i \underline{f}$ *is a monomorphism, and* $\Gamma \pi_i \underline{f}$ *is a monomorphism.*

(4.1.4) REMARK. These theorems are the exact analogues of Dror's results (see [**DR**, Theorem (3.1) and Proposition (3.2)]).

(4.2) *Some calculations*. Suppose $\underline{X}$ is an object of $\mathcal{C}$. It is easy to see that for any i, $1 \leqslant i < \infty$, we have $\Gamma_\omega \pi_i \underline{X} \approx \Gamma'_\omega \pi_i \underline{X} \approx \Gamma \pi_i \underline{X} \approx 0$ in any of the following cases:

(4.2.1) $\pi_1 \underline{X} \approx 0$;

(4.2.2) $\underline{X}$ is simple (see (3.2.3) for a definition);

(4.2.3) $\underline{X}$ is nilpotent (see (3.2.3) for a definition);

(4.2.4) $\underline{X}$ is complete (see (3.2.4) for a definition).

(4.3) A COROLLARY OF THEOREM (4.1.3). *Given a map* $\underline{f}\colon \underline{X} \to \underline{Y}$ *in* $\mathcal{C}$. *Suppose any one of the following holds*: $\pi_1 \underline{X} \approx \pi_1 \underline{Y} \approx 0$; $\underline{X}$ *and* $\underline{Y}$ *are simple*; $\underline{X}$ *and* $\underline{Y}$ *are nilpotent*; *or* $\underline{X}$ *and* $\underline{Y}$ *are complete. Then the following are equivalent*:

(i) $\underline{f}$ *is a pro-homology equivalence.*

(ii) $\underline{f}$ *is a weak pro-homotopy equivalence.*

(4.4) $\underline{H}$*-structures in* $\mathcal{C}$. The notion of an H-space or a space with an H-structure is well known in homotopy theory; moreover, Eckmann and Hilton [**EH**] have discussed an analogous notion of $\underline{H}$-structure on objects of suitable categories. In our earlier work [**SI$_2$**], we have concretized the notion of $\underline{H}$-structure in suitable pro-homotopy categories; indeed, it follows from Theorem (4.2.1) of [**SI$_2$**] that an object $\underline{X}$ of pro-$\mathcal{H}\mathcal{C}\mathcal{W}_0$ with an $\underline{H}$-structure is simple.

(4.5) *Shape theoretic considerations.* A pointed topological space X is called *s-simple* ("shape simple"), *s*-nilpotent, or *s*-complete if there exists an object $\underline{X}$ of pro-$\mathcal{HCW}_0$ associated with X in the sense of Morita (cf. **[MA]**, **[DS]**) such that $\underline{X}$ is simple, nilpotent, or complete; see (3.2). We shall be brief; see **[SI$_2$]** for a discussion of many related matters. There are many versions of the Whitehead theorem in shape theory, for instance, **[MA]**, **[MO]**, **[MR]**; moreover, the books **[BO]**, **[DS]**, **[ED]** contain many other references and related discussions. As a sample, we shall next state an analogue of the Dror-Whitehead theorem in shape theory.

(4.5.1) THEOREM. *Suppose $\underline{f}$: $X \to Y$ is a shape map (or a shaping) of pointed continua of finite fundamental dimension and suppose X and Y are s-simple or, more generally, s-nilpotent. Then the following are equivalent:*

(a) *$\underline{f}$ is a shape equivalence.*

(b) *$\underline{f}$ induces isomorphisms of all the homotopy pro-groups.*

(c) *$\underline{f}$ induces isomorphisms of the homology pro-groups with coefficients in $\mathfrak{Z}$.*

PROOF. The equivalence of (a) and (b) is well known (cf. **[DS]**). It follows from Corollary (4.3) that (c) implies (b). This proves the theorem. □

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